Reader and Student Notes on General Relativity

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Based on The Theoretical Minimum Vol. IV dedicated to GR by Prof. Susskind and Andrè Cabannes

***Abstract*—Notes—a kind of summary—to help the author of this doc to both consolidate what I've learned and have a quick reference for the main ideas and concepts of GR whenever I need a refresher. Based on the excellent job of Prof. Susskind and Andrè Cabannes. I have always believed that to write meaningfully about a topic, one must be an expert and have spent many hours developing a deep understanding of its many interconnections. At the same time, I also believe that writing is one of the best ways to achieve that depth of understanding. For that reason, I recommend turning to professional and expert academic sources to learn about this subject and treating this document merely as a set of student notes—which may well contain errors.**

I. Flatness and Gravity

THE problem of determining whether we are in the presence of a gravitational field in a region is mathematically equivalent to the problem of finding out whether the geometry of that region is flat or not. E. Let's consider a 3D space and let's take a surface $\mathbf{S}$ with each point located by means of two coordinates X^1 and X^2. This surface has a metric to define infinitesimal lengths by

$$dS^2 = g_{mn}(\mathbf{X})dX^m dX^n \tag{1}$$

where m and n run from 1 to 2. This expression follows the Einstein sumation convention. A flat geometry requires the following

$$d\mathbf{S}^2 = \delta_{m,n}dX^m dX^n = (dX^m)^2 + (dX^n)^2 \tag{2}$$

where $\delta_{m,n}$ refers to the delta Kronecker function. If the metric we have does not satisfy this requirement, that does not mean that the surface is not flat, as it might mean that the system of coordinates we are using to which this metric belongs, is not flat. But we might find another system of coordinates Y such that at any point *P*, a located by those Y coordinates, the metric has the form from eq. 2. as follows once we transform from X to Y.

$$d\mathbf{S}^2 = \delta_{m,n}dY^m dY^n = (dY^m)^2 + (dY^n)^2 \tag{3}$$

The technique to find this system of coordinates is to search for a diagnostic quantity, built out of the metric and its derivatives that then we can compute. That diagnostic quantity is the *curvature tensor*, or the *Gaussian curvature* in two dimensions. If it is zero everywhere then the space is flat. If it is not at any location, then the space shows some curvature.

We need to find a system of coordinates such that we force the metric in that system to look like that of eq. 2.

Theorem 1.1: At any given point P in the space, we can find a system of coordinates in which the metric is $\delta_{m,n}$ to first order in small deviations from the point. In general, unless the space is flat, the attempt will fail beyond first order.

These coordinates are called *Gaussian Normal Coordinates*. It consists of a tangent plane to the surface S at point P, whose axes are perpendicular to each other. At every point P of the surface you can choose Gaussian Normal Coordinates such that, at that point, whose coordinates are X_0, we have

$$g_{mn}(X_0) = \delta_{mn} \tag{4}$$

The theorem says that once you have chosen the directions at point P, you can also choose the X's such that the derivative of any element of the metric tensor g_{mn} at that point with respect to any direction in space, X^r, can be set equal to zero. We consider P at the origin of the new coordinate system X

$$\frac{\partial g_{mn}}{\partial X^r} = 0 \tag{5}$$

Unless the space is flat, the higher order derivatives won't be zero

$$\frac{\partial g_{mn}}{\partial X^r \partial X^s} \neq 0 \tag{6}$$

The reason has to do with the Taylor expansion of each coordinate X^m in terms of Y and also the expansion of the metric tensor. We have enough degrees of freedom to select our coordinate system so that the first order derivative of the expansion of g_{mn} is zero. See appendix for details. So we can construct coordinates X's such that P is located at the origin and the metric tensor has the form

$$g_{mn}(X) = \delta_{mn} + o(X) \tag{7}$$

where o(X) represents the second order and higher. That means that spaces are Euclidean up to a second order.

Let's say we have a vector V on a surface S with a system of coordinates Y and two points P, or origin, and Q, as in the following figure. If we want to move this vector from $P \rightarrow Q$, then we can differentiate each component, that is, the covariant component V_m over the r direction and afterwards over the s direction. Or the other way around.

$$D_r D_s V_m - D_s D_r Vm \tag{8}$$

That is equivalent to a second-order partial derivative of V_m, expressed in covariant derivatives of a tensor. It is also equivalent to moving the vector from P to Q along two different paths. If eq. 8 is zero, then the space is flat, as in curved space it is not the same to follow one path or another to go the same point Q from P.

In flat space covariant derivatives are interchangeable. In curved space they are not. This allows us to test whether a space is flat or not. We will test whether differentiating tensors and in particular vectors in opposite order gives the same result:

- If the answer is yes everywhere in the space for any vector, then the space is flat.
- If we discover that there are places in the space where the order of differentiation gives different answers, then we know that the space has some kind of defect in it.

Covariant Derivatives

The ordinary derivatives of the components of a vector with respect to the coordinates do not themselves form a tensor, because the derivative depends on the coordinate system selected (see figure 3 and 4 in lesson 3). Hence, we need a better way for the derivative of a vector than just differentiating its components, something that if it is zero in one frame is zero in every other frame of reference. The reason is that given a space with a coordinate system, we might want to find another coordinate system such that we can test if the metric tensor has only first order term or not, which is equivalent to a flat space or not. We saw that one way of testing so is by differentiating the vector following one direction or the other (different order of partial derivatives). If the result is the same, then it is flat. We do not want that the result of differentiating our vector is different depending on the coordinate system. Therefore, the differentiation needs to produce a tensor to avoid that. Here is a valid new way to achieve that: We have a vector field V on a space equipped with any coordinate system Y. We want to calculate the derivative of V at a point P. Then follow these steps:

1) Change coordinates to use Gaussian normal coordinates at P, let's call them X (they are approximately flat at P and valid over the whole surface).
2) Differentiate V at P in the usual way, using X-coordinates
3) Consider the collection of partial derivatives we got as the components of a tensor fo rank 2 in the X-coordinates
4) Switch back to our original coordinate system Y, and re-express in that original system the tensor we got, using the tensor equations linking X and Y.

With that definition, the derivatives at P are the old vector ones, corrected by the addition of a term:

$$D_r V_m = \frac{\partial V_m}{\partial Y^r} - \Gamma^t_{rm} V_t \tag{9}$$

Where the term on the left is the partial derivative of V_m with respect to the r direction in X. Each member of the equation by itself is not a tensor. But the result is a tensor. Γ is the connection coefficient or Christoffel symbols. They connect neighboring points and tell us how to calculate the rate of change of a vector field from one point to another nearby, even though the coordinate system may be changing. They are symmetric. To differentiate a two index tensor, we just add more indexes:

$$D_r T_{mn} = \frac{\partial T_{mn}}{\partial Y^r} - \Gamma^t_{rm} T_{tn} - \Gamma^t_{rn} T_{mt} \tag{10}$$

Important to note that the Christoffel symbols are related to the coordinate system, not to the intrinsic geometry of the space.

$$\Gamma^t_{mn} = \frac{1}{2} g^{rt} [\partial_n g_{rm} + \partial_m g_{rn} + \partial_r g_{mn}] \tag{11}$$

According to that, we can rewrite

$$D_s D_r V_n = D_s [\partial_r V_n - \Gamma^t_{rn} V_t] \tag{12}$$

If we continue to differentiate and replace:

$$D_s D_r V_n - D_r D_s Vn = R^t_{srn} V_t \tag{13}$$

Where:

$$R^t_{srn} = \partial_r \Gamma^t_{sn} - \partial_s \Gamma^t_{rn} - \Gamma^p_{sn} \Gamma^t_{pr} - \Gamma^p_{rn} \Gamma^t_{ps} \tag{14}$$

The tensor R is the curvature tensor, or Riemann curvature tensor. Given that there partial derivatives of the connection coefficients, this is equivalent to second order partial derivatives of the metric tensor in eq. 11, so if the space is flat, then the Riemann tensor will be zero as the first order are surely zero and also the second order also. If it is not, then the space is not flat. The quantity in gravitation to which the curvature tensor is related is the local quantity known as the *Tidal Forces*. That is, tidal forces are represented by the curvature tensor.

In a Gaussian normal coordinate system, the metric tensor is locally the Kronecker-delta tensor up to second order. Therefore, in any given point P, in a set of Gassian normal coordinates at that point, the ordinary partial derivative of first order of the components of the metric tensor is zero:

$$\partial_r g_{mn} = 0 \tag{15}$$

According to how we have defined it, the covariant derivative of the metric tensor in any coordinate system at P on the surface is equal to zero:

$$D_r g_{mn} = 0 \tag{16}$$

According to 16 and 11, the Christoffel symbols are zero in any coordinate system whose ordinary partial derivatives of the metric tensor are zero.

Contravariant Derivatives

We can write the contravariant form of a vector as:

$$V^m = g^{mp} V_p \tag{17}$$

Then we can take the covariant derivative on both sides y aplicamos la regla de la cadena:

$$D_r V^m = D_r g^{mp} V_p + g^{mp}(D_r V_p) = g^{mp}(D_r V_p) \tag{18}$$

as the covariant derivative of the inverse of the metric tensor must also be zero as the covariant of the derivative of the metric tensor is zero. Now we plug equation 9 and after some manipulation the result is:

$$D_r V^m = g^{mp}(D_r V_p) = \partial_r V^m + \Gamma^m_{rt} V^t \tag{19}$$

Parallel Transport

We are interested in knowing, when we move along a curve on the space, whether the field stays parallel to itself. At each point of the curve, parallelo to itself between X and X + dX on the curve means that its covariant derivative in the direction of the curve at that point X is 0. That derivative along the trajectory corresponds to taking the covariant derivative and multiplying by dX, in each component:

$$DV^n = D_m V^n dX^m = \frac{\partial V^n}{\partial X^m} dX^m + \Gamma^n_{mr} V^r dX^m \quad (20)$$

$$DV^n = dV^n + \Gamma^n_{mr} V^r dX^m \quad (21)$$

Then a vector remains parallel along a trajectory if:

$$DV^n = dV^n + \Gamma^n_{mr} V^r dX^m = 0 \quad (22)$$

A parallel transport is independent of the coordinate system. Considering a curve S and a distance dS between the two points X and X + dX, defined by:

$$dS^2 = g_{mn} dX^m dX^n \quad (23)$$

The way we construct the tangent vector in the X coordinate system is by dividing over dS the corresponding derivative of the component:

$$t^m = \frac{dX^m}{dS} \quad (24)$$

Then, if the tangent vector remains parallel to itself along the trajectory, the following condition holds:

$$Dt^n = dt^n + \Gamma^n_{mr} t^r dX^m = 0 \quad (25)$$

This equation can be written as follows by dividing over dS:

$$\frac{dt^n}{dS} = -\Gamma^n_{mr} t^r \frac{dX^m}{dS} = -\Gamma^n_{mr} t^r t^m \quad (26)$$

We can write the equation as a second derivative, which resembles to an acceleration:

$$\frac{d^2 X^n}{dS^2} = -\Gamma^n_{mr} t^r \frac{dX^m}{dS} = -\Gamma^n_{mr} t^r t^m \quad (27)$$

A particle in a gravitational field moves along the straightest possible trajectory, both through space and time.

Uniform Acceleration

In figure 1, we see the light cone and the hyperbolic trajectory of a uniformly accelerated particle for one spatial dimension and the time dimension. We consider that trajectory the locus of all points a fixed space-like Minkowski distance from the origin in that figure with the form of a hyperbola:

$$X^2 - T^2 = r^2 \quad (28)$$

This form is hyperbolic because in special relativity the maximum velocity this particle can achieve is the speed of light. Therefore, a uniformly accelerated observer is one moving on a hyperbola as shown in 1. We can write the coordinates of a point P along the trajectory, i.e. $P_1(\omega_1, r)$, as:

$$X = r \cosh \omega \quad (29)$$

$$T = r \sinh \omega \quad (30)$$

$$\omega = \frac{\tau}{r} \quad (31)$$

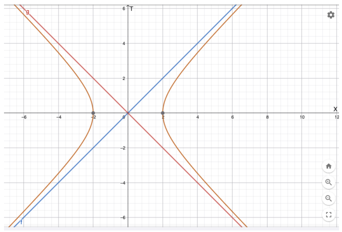

Fig. 1: Light cone

where τ is the proper time measured along the trajectory. r is like a space coordinate and ω is like a time coordinate. In figure 2 we see several trajectories corresponding to *r = 1*, *r = 2*, *r = 3*. At fixed values of ω, the distances between the observers corresponding to those *r's* are always the same:

$$|MN| = |NP| = |RS| = |ST| \quad (32)$$

As the Lorentz frame of reference accelerates, the distance between neighboring observers stays the same. However, one can see that the accelerations along the different trajectories are different from a non-relativistic accelerated frame of reference. Accelerations are different between them in spite they distance remains the same. The relativistic acceleration

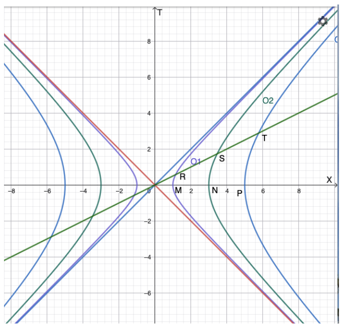

Fig. 2: Light cone

is a 4-vector defined by:

$$a^\mu = \frac{d^2 X^\mu}{d\tau^2} \quad (33)$$

The acceleration is constant along a hyperbolic trajectory because the proper length of a^μ is constant:

$$|a|^2 = (a^1)^2 - (a^0)^2 = constant \quad (34)$$

where a^0 refers to T and a^1 refers to X. We can write:

$$X = r \cosh \frac{\tau}{r} = a^1 \quad (35)$$

$$T = r \sinh \frac{\tau}{r} = a^0 \quad (36)$$

$$|a|^2 = r^2 \; (\cosh^2 \frac{\tau}{r} - \sinh^2 \frac{\tau}{r}) = r^2 \quad (37)$$

$$|a| = \frac{1}{r} \quad or \quad |a| = \frac{c^2}{r} \quad (38)$$

The second expression for $|a|$ introduces de speed of light.

Let's suppose that we have an elevator at a distance y of R that is accelerated along the hyperbola, almost vertical

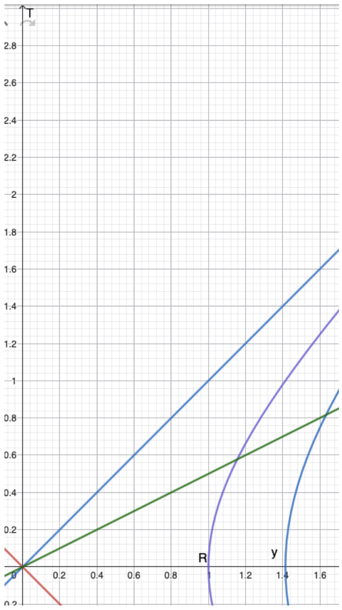

Fig. 3: Elevator at distance y from R

at this small distance that will travel during an small amount of time, as shown in figure 3. In ordinary polar coordinates in Euclidean plane, the hyperbolic trajectory can be expressed as:

$$d\tau^2 = r^2 d\omega^2 - dr^2 \tag{39}$$

$$r = y + R \tag{40}$$

$$dr = d(y + R) = dy \tag{41}$$

$$d\tau^2 = (1 + \frac{2y}{R} + \frac{y^2}{R^2})\ R^2\ d\omega^2 - dy^2 \tag{42}$$

We focus on a very small region around R, then, as y/R is small and $R\omega = t$, we can simplify:

$$d\tau^2 = (1 + \frac{2y}{R})\ dt^2 - dy^2; \tag{43}$$

$$d\tau^2 = (1 + 2gy)\ dt^2 - dy^2 \tag{44}$$

where we make use of $g = 1/R$ (remember $g = a$ acceleration), with gy being the gravitational potential. Therefore, in that elevator accelerating without the presence of gravitational field created by another body, we find there is, however, an effective gravitational field (fictitious gravitational field). Free particles moves in geodesics of space-time, then we can use equation 27, together with $dS = i d\tau$, and write:

$$\frac{d^2X^n}{dS^2} = \frac{d^2y}{d\tau^2} = -\Gamma^y_{mr}\frac{dX^r}{d\tau}\frac{dX^m}{d\tau} \tag{45}$$

If we do the calculations and considering that $\frac{dX^0}{d\tau} = 1$ and that the velocity is small compare to the speed of light:

$$\frac{d^2y}{d\tau^2} = -\Gamma^y_{tt} = \frac{1}{2}\frac{\partial g_{tt}}{\partial y} \tag{46}$$

From 43 we know that the component g_{tt} of the tensor g must be $1 + 2gy$, so the derivative with respect to y drives to:

$$\frac{d^2y}{d\tau^2} = -\Gamma^y_{tt} = \frac{1}{2}(-2g) = -g \tag{47}$$

We now know that:

- Space-time has a metric. In uniformly accelerated coordinates, it is almost the Minkowski metric with the extra term $2gy$.
- The equation of motion along a geodesic of space-time (travelling slowly) is Newton's equation in a uniform gravitational field.
- We can guess that the effect of a real gravitating matter will be $-G/y$, instead of $2gy$.

$$d\tau^2 = (1 - \frac{2GM}{y})\ dt^2 - dy^2 \tag{48}$$

where y refers to the horizontal axis.

Geodesics and Euler-Lagrange Equations

A geodesic is a curve whose tangent vector stays parallel to itself all along the curve. In a geodesic the covariant derivative of the tangent vector is everywhere zero. Another way to get to the equation of a geodesic is by making minimun the length of the curve between two points, finding the shortest path. Therefore, the quantity to minimize is the proper time between points 1 and 2 in space-time, given by:

$$\tau = \int_1^2 \sqrt{-g_{\mu\nu}(X)dX^\mu dX^\nu} \tag{49}$$

A time-like geodesic maximizes proper time, so given the usual definition of the action, this needs to be proportional to minus the proper time:

$$action = -m\int d\tau \tag{50}$$

That is equivalent to:

$$action = -m\int_1^2 \sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}}dt \tag{51}$$

We know that an Action is by definition equal to the integral of the Lagrangian, which is function of the velocities and position $\mathcal{L}(\dot{X}, X)$. In this case, the Lagrangian is:

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{dt}\frac{dX^\nu}{dt}} \tag{52}$$

Schwarzschild Metric

We are interested now on the question of the metric of space-time and the motion of a particle in a real gravitational field of a massive spherically symmetric object with mass M at a distance X of a particle. We start with the metric of a flat space given by:

$$d\tau^2 = dt^2 - \frac{1}{c^2}dX^2 \tag{53}$$

As the gravitating object warps the space-time, we need to add a coefficient in front of dt^2 only, which represents the gravitational potential:

$$d\tau^2 = (1 + \frac{2U(x)}{c^2})\ dt^2 - \frac{1}{c^2}dX^2 \tag{54}$$

Including this in the formula of the Action and after doing some arrangements:

$$action = -mc^2 \int \sqrt{(1 + \frac{2U(x)}{c^2}) - \frac{\dot{X}^2}{c^2}} dt \quad (55)$$

We can simplify using the binomial theorem:

$$action = \int (-mc^2 - mU + \frac{m}{2}\dot{X}^2)\ dt \quad (56)$$

That is valid when the speed of the particle is small compared to the speed of light. We are making here a non-relativistic aproximation. Note that:

$$-g_{00} = 1 + \frac{2U(x)}{c^2} \quad (57)$$

The gravitational potential energy created in space by a body of mass M is:

$$U(X) = -\frac{MG}{r} \quad (58)$$

In polar coordinates, the proper time is:

$$d\tau^2 = (1 - \frac{2MG}{c^2 r})\ dt^2 - \frac{1}{c^2} dr^2 - \frac{1}{c^2} r^2 (d\theta^2 + cos^2\theta d\phi^2) \quad (59)$$

According to Eistein's field equations that we will see later, we need to add another factor in front of dr^2, hence the result is:

$$d\tau^2 = \left(1 - \frac{2MG}{c^2 r}\right) dt^2 - \frac{1}{1 - \frac{2MG}{c^2 r}} \frac{1}{c^2} dr^2 - \frac{1}{c^2} r^2 \left(d\theta^2 + \cos^2\theta\, d\phi^2\right) \quad (60)$$

Let's suppose a rocket that travels in a radial direction straight to the mass M, so that we can disregard the spherical coordinates and use only the radial. The action is as follows after dividing inside the square root by dt^2 and outside the square root by dt:

$$A = -m \int \sqrt{\left(1 - \frac{2MG}{r}\right) - \frac{1}{1 - \frac{2MG}{r}} \frac{dr^2}{dt^2}}\ dt \quad (61)$$

Therefore, the Lagrangian is:

$$\mathcal{L} = -m \sqrt{\left(1 - \frac{2MG}{r}\right) - \frac{1}{1 - \frac{2MG}{r}}\ \dot{r}^2} \quad (62)$$

If we calculate the Hamiltonian and put the velocity of the rocket int terms of the Hamiltonian, we obtain:

$$\dot{r}^2 = \left(1 - \frac{2MG}{r}\right)^2 - \frac{\left(1 - \frac{2MG}{r}\right)^3}{E^2} \quad (63)$$

Then, the following conclusion can be obtained: As the distance of the rocket to the mass tends to $r \to 2MG$, the velocity of the rocket as seen by the observer tends to zero. Also, at that distance the proper time changes to negative. That is the Schwarzschild radius of a Black Hole. For the observer in the rocket, nothing happens when it get to that radius, however, the external observer sees the rocket as going increasingly more slowly. This takes place only for bodys whose masses are contained within its Schwarzschild radius, that is 9 mm. for Earth and 3 km. for the Sun.

Newton's Field Equations

It is known that a field affects particles as follows:

$$F = -m\ \nabla\phi(x) \quad (64)$$

On the other hand, masses in space tell the gravitational field what to be, following the Poisson's equation:

$$\nabla^2\phi(x, y, z) = 4\pi G\rho(x, y, z, t) \quad (65)$$

In this equation 4π is a convention originated from the geometry of the sphere, as we assume that we are dealing with a field that is invariant under rotation. G is Newton's Gravitational constant and $\rho(x, y, z, t)$ is the density of mass at each position and time. In what follows, we are going to consider a body of mass M We impose the only constraint that the density of mass in the object of total mass M must be symmetric with respect to rotation, that is, spherical symmetry. For example, more density in the center than in outer layers of the object is fine, provided that it only depends on the radial coordinate r and not on the angular coordinates ϕ or θ. Therefore, according to Newton's theorem, the object is equivalent to a point of mass M located at the center of the object.

Energy-Momentum Tensor

Massless particles, such as a photon, or massive particles, such as neutrons, have both energy and momentum. Therefore, energy and matter have these properties. We can ask how much momentum and energy exists in a certain volume, in the form of particles or energy, including the mc^2 part of energy. They can be described in terms of density of energy and density of momentum. Besides that, energy and momentum are conserved quantities. The conservation of energy and momentum are local properties. They cannot appear or disappear, they flow across the boundaries of a region. When an object is moving, the energy and momentum is flowing too. Energy and Momentum together form a 4-vector. Total energy and momentum are not invariant. For example, if a particle is still and I am moving, I say the particle has momentum and kinetic energy, not only mc^2 energy. But if I am still, then I see no momentum or kinetic energy added to mc^2. Also, due to Lorentz contraction, the volume of space depends on the reference frame and so, the density. Hence, density and flow are not invariants. We define the following 4-vector:

$$P^\mu = (E, P^m) \quad (66)$$

The quantity E is the energy, also called P^0. And the P^m are the components of the momentum. Their components change according to tensor equations of change of reference frame. Also the total quantities change, they are not invariant. Each of the four components of P^μ is a conserved quantity in the sense of invariance of 4-vector, that is, if two 4-vectors are equal in on frame, they are equal in every frame. Let's define now a tensor $T^{\mu\nu}$, in which the superindexes:

- $\mu = 0$ refers to energy and $\mu = 1, 2, 3$ refers to momentum.
- $\nu = 0$ refers to density of energy or momentum and $\nu = 1, 2, 3$ refers to flow of energy of momentum.

For example, $T^{00}(x, y, z)$ refers to the density of energy in the point (x, y, z), and $T12(x, y, z)$ refers to the flow of the y's component of the momentum in (x, y, z). As energy and momentum are conserved quantities, the $\nu > 0$ components of the tensor represent the flow of these quantities through a unit surface along each direction x, y, z of the point. Therefore, the continuity equation for energy and momentum can be written as:

$$\frac{DT^{\mu\nu}}{DX^{\nu}} = 0 \tag{67}$$

This matrix T is called the *energy-momentum tensor*. This matrix has the property that it is symmetric.

Ricci Tensor and Curvature Scalar Tensor

The Ricci Tensor $R^{\mu\nu}$ is a contraction of the Riemann tensor $R^{\sigma}_{\mu\nu\tau}$. We start with the fully covariant Riemann tensor:

$$R_{\rho\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\rho} g_{\lambda\sigma} - \partial_\nu \Gamma^\lambda_{\mu\rho} g_{\lambda\sigma} + \Gamma^\lambda_{\mu\kappa} \Gamma^\kappa_{\nu\rho} g_{\lambda\sigma} - \Gamma^\lambda_{\nu\kappa} \Gamma^\kappa_{\mu\rho} g_{\lambda\sigma} \tag{68}$$

Then we convert the Riemann tensor to the mixed form $R^{\rho}_{\sigma\mu\nu}$ using the inverse metric:

$$R^{\rho}_{\sigma\mu\nu} = g^{\rho\lambda} R_{\lambda\sigma\mu\nu} \tag{69}$$

After this, we contract the first and third indices (set $\rho = \mu$) to obtain the Ricci tensor:

$$R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu} = \sum_{\rho} g^{\rho\lambda} R_{\lambda\sigma\rho\nu} \tag{70}$$

We need to relabel $\sigma \to \mu$ to match the conventional notation for the Ricci tensor:

$$R_{\mu\nu} = \sum_{\rho,\lambda} g^{\rho\lambda} R_{\lambda\mu\rho\nu} \tag{71}$$

The curvature scalar R then is a contraction:

$$R = R^{\mu\tau} g_{\rho\tau} \tag{72}$$

Einstein's field equations

Finally we arrive to the Einstein's field equations:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu} \tag{73}$$

They generalize Newton's equations and in the appropriate limit, they reduce to Newton's equations. The show how the source of the gravitational field is not just energy density, it can involve energy flow, momentum density and momentum flow. In not relatisvistic scenarios, energy density is the biggest component, being the rest smaller.

References

[1] Susskind, Leonard, Cabannes, André. General Relativity. The theoretical minimum. Penguin House, 2023.